Pulse-Echo Ultrasound Imaging Combining Compressed Sensing and the Fast Multipole Method

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Pulse-Echo Ultrasound Imaging Combining Compressed Sensing and the Fast Multipole Method

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Abstract—We introduced the fast multipole method (FMM) into our concept for plane wave pulse-echo ultrasound imaging (UI) to reduce the memory consumption and the computational costs associated with the numerical solution of the underlying regularized linear inverse scattering problem (ISP). For an example of typical size and in comparison to the conventional approach, we showed that the FMM requires less than 0.25% of the memory and less than 24% of the number of complex-valued multiplications. The FMM thus enables the numerical solution of the regularized (e.g. by compressed sensing) linear ISP on standard personal computers. It significantly improves the applicability of inverse scattering strategies in practical UI.

I. INTRODUCTION

Modern concepts for pulse-echo ultrasound imaging (UI), e.g. coherent plane wave compounding [1] or synthetic aperture focusing [2], are usually based on physical models that, assuming homogeneous media, exclusively account for the ultrasound waves’ times-of-flight. The object’s inhomogeneous mechanical properties, e.g. spatial fluctuations in compressibility or mass density, the associated types of scattering, diffraction, and frequency-dependent absorption and dispersion are neglected.

In contrast, the physical description of pulse-echo UI as a linear inverse scattering problem (ISP) in the temporal frequency domain can readily account for these effects [3], [4]. In combination with novel regularization procedures, e.g. compressed sensing (CS) [5], the formulation as linear ISP significantly improves image quality [6], [7]. Since all temporal frequency components can be treated independently, the frequency domain formulation is well suited for parallel computer architectures. However, memory consumption and computational costs owing to complex-valued arithmetic and huge problem sizes are major challenges.

In this contribution, we demonstrate that these challenges can be met effectively by the usage of the fast multipole method (FMM) [3]. The FMM is based on a multipole expansion of the free-space Green’s function associated with the Helmholtz equation. Instead of independently computing the scattered sound field for each scatterer, the expansion enables the rapid computation of the scattered sound field originating from groups of scatterers.

The paper is organized as follows. In Section II we briefly review the discretized linear ISP underlying plane wave pulse-echo UI presented in preceding publications [3], [4] and its regularization using CS. In Section III we introduce the FMM into the ISP and outline an efficient algorithm to compute the scattered sound field. The reductions in memory consumption and computational costs are quantified in Section IV.

II. THE LINEAR INVERSE SCATTERING PROBLEM IN THE TEMPORAL FREQUENCY DOMAIN

The underlying physical model was introduced in detail in preceding publications, e.g. [3], [6], [7]. In the following, only a brief review of the discretized linear ISP will be provided.

The scan configuration typically used in two-dimensional pulse-echo UI is illustrated in Fig. 1. An inhomogeneous object \( \Omega \subset \{ (x, z)^T \in \mathbb{R}^2 : z > 0 \} \) (gray region) with compressibility \( \kappa_1 : \Omega \rightarrow \mathbb{R} \) is surrounded by a homogeneous fluid with compressibility \( \kappa_0 \in \mathbb{R} \). For \( r \in \mathbb{R}^2 \), the relative spatial fluctuation in compressibility \( \gamma_c \) is defined as

\[
\gamma_c(r) = \begin{cases} 
1 - \kappa_1(r)\kappa_0^{-1} & \text{for } r \in \Omega, \\
0 & \text{for } r \notin \Omega.
\end{cases}
\]  

Fig. 1. Typical scan configuration employed in two-dimensional pulse-echo ultrasound imaging.

The spatial discretization of the ISP is based on the regular lattice

\[
\mathcal{L} = \{ r_i \in \mathbb{R}^2 : r_i = r_0 + i x \mathbf{e}_x + i z \mathbf{e}_z, \quad 0 \leq i_x < N, 0 \leq i_z < N, i = i_x N + i_z \} 
\]  

and the set of virtual transducer elements

\[
\mathcal{V} = \{ r_{\text{vir}, v} \in \mathbb{R}^2 : r_{\text{vir}, v} = (x_0 + v\zeta)\mathbf{e}_z, 0 \leq v < N \},
\]  

where \( r_0 = (x_0, z_0)^T = (-2^{-1}(N-1)\zeta, z_0)^T \), \( z_0 > 0 \), denotes an offset vector, \( \mathbf{e}_x \) and \( \mathbf{e}_z \) indicate the unit vectors in the directions of the positive coordinate axes, and \( N = 2^{L_0}, L_0 \in \mathbb{N} \).
The broadband sound wave emitted by the transducer array is scattered by the fluctuations (1). Considering monofrequent perturbations of discrete frequency \( f_l \in \mathbb{R}^+ \), \( l \in \mathbb{N}_0 \), and using the Born approximation, the scattered acoustic pressure \( p_{sc}^\ell \) is given by the system of linear equations

\[
p_{sc}^\ell (r_{vir,v}, e_{\vartheta,d}) = (k_l c_0^{-1})^2 \sum_{i=0}^{N^2-1} \gamma_{\ell, BP}(r_i) p_l^0(r_i, e_{\vartheta,d}) \times g_l(r_{vir,v} - r_i), \tag{4}
\]

where \( k_l = 2\pi f_l c_0^{-1} \) is the wavenumber, \( c_0 = (\kappa_0 \rho_0)^{-\frac{1}{2}} \) is the small-signal sound speed in the homogeneous medium, \( \rho_0 \) is the homogeneous mass density of both media, \( \gamma_{\ell, BP}(r_i) \), \( 0 \leq i < N^2 \), are samples of the bandpass-filtered fluctuation (1) observed by the UI device, and

\[
p_l^0 (r, e_{\vartheta,d}) = P_l(e_{\vartheta,d}) e^{-j k_l c_0^{-1} r} \tag{5}
\]

is an incident plane wave with frequency-dependent amplitude \( P_l \in \mathbb{C} \) and direction of propagation \( e_{\vartheta,d}, 0 \leq d < N_\vartheta \). Further,

\[
g_l(r) = \frac{j}{4} H_0^{(2)} (k_l \|r\|_2) \tag{6}
\]

denotes the free-space Green’s function satisfying the Sommerfeld radiation condition [9] and \( (\Delta + k_l^2) g_l(r) = \delta(r), \|r\|_2 \) is the \( \ell_2 \)-norm of \( r \), \( H_0^{(2)} \) denotes the zero-order Hankel function of second kind, and \( \delta \) is the Dirac delta distribution. The scattered acoustic pressure in the time domain \( p_{sc}^\ell \) is related to \( p_{sc}^\ell \) via the identity \( p_{sc}^\ell (r, e_{\vartheta,d}, t) = \Re \{ p_{sc}^\ell (r, e_{\vartheta,d}) e^{j 2\pi f_l t} \} \).

Considering multiple discrete frequencies \( f_l \), \( 0 \leq l < N_f \), and \( N_\vartheta \) directions of the incident plane wave \( \{5\} \), the system of linear equations \( \{4\} \) can formally be written as \( \{3\} \).

\[
\mathbf{p}^\ell = \mathbf{G} \mathbf{\gamma}_{n, BP} \tag{7}
\]

where \( \mathbf{G} \) denotes a complex-valued \( N_\vartheta N_f N \times N^2 \) matrix and \( \mathbf{\gamma}_{n, BP} \) indicates the \( N^2 \times 1 \) vector with the values \( \gamma_{n, BP}(r_i) \).

The objective in pulse-echo UI is to recover the vector \( \mathbf{\gamma}_{n, BP} \) from the measurements \( \mathbf{p}^\ell \). To enable the usage of the CS formalism for the regularization of the ill-posed system (7), we assume that there exists an orthonormal linear transform \( \mathbf{\Psi} \), e.g. the Fourier or a wavelet transform, such that \( \mathbf{\gamma}_{n, BP} = \mathbf{\Psi} \mathbf{\theta} \) with a nearly sparse coefficient vector \( \mathbf{\theta} \). We further assume that the matrix \( \mathbf{G} \) has the required properties [3]. The relative spatial fluctuations \( \mathbf{\gamma}_{n, BP} \) can then be recovered by solving the \( \ell_1 \)-minimization problem

\[
\hat{\mathbf{\theta}} = \arg \min_{\mathbf{x} \in \mathbb{C}^{N^2}} \| \mathbf{x} \|_1 \text{ s.t. } \| \mathbf{G} \mathbf{\Psi} \mathbf{x} - \mathbf{p}^\ell \|_2 \leq \epsilon \tag{P0}
\]

where \( \| \mathbf{x} \|_1 = \sum_{i=0}^{N^2-1} |x_i| \) is the \( \ell_1 \)-norm of \( \mathbf{x} \) and \( \epsilon > 0 \) is a measure for noise and inaccuracy of the physical model.

The optimization problem (P0) is usually solved numerically. The available algorithms compute matrix-vector products involving the matrix \( \mathbf{G} \mathbf{\Psi} \) or its adjoint \( \mathbf{\Psi}^H \mathbf{G}^H \) in an iterative-valued. Since \( \mathbf{G} \) is densely occupied, complex-valued, and usually very large (see Section [11] for an analysis), this procedure is afflicted with both high memory consumption and computational costs. The FMM meets these challenges by partially decomposing the matrix \( \mathbf{G} \) into diagonal matrices.

### III. The Fast Multipole Method (FMM)

The FMM was introduced by Rokhlin [8] in 1990 as a tool for the rapid solution of integral equations. The FMM is based around an approximate representation of the Green’s function (6) in (4) that is valid under a specific geometrical relationship of the source’s location \( r_i \) and the sink’s location \( r_{vir,v} \). For this reason, both the lattice (2) and the virtual transducer elements (3) have to be partitioned into subsets, which will be referred to as groups in the sequel.

The groups’ geometrical relationships will be classified next.

#### A. Group Geometry

The lattice (2) is partitioned into \( 2^{2L} \) square-shaped and disjoint groups, where the level parameter \( L \), \( 1 \leq L < L_0 \), determines the number of point scatterers per group. For a given level \( L \) and \( 0 \leq m, n < 2^L \), the groups are denoted by \( \ell_{m,n}^{(L)} \). The groups’ center coordinates are indicated by \( r_{\ell,m,n}^{(L)} \), and the identical length of the groups’ diagonals is \( d^{(L)} \). Group-internal locations are denoted by \( r_{\ell'} = r_{\ell} - r_{\ell, lat,m,n}^{(L)} \), \( 0 \leq \ell' < (2^{-L} N)^2 \). They are identical for all groups \( \ell_{m,n}^{(L)} \).

Similarly, the set of virtual transducer elements (3) is partitioned into \( 2^L \) disjoint groups. For a given level \( L \) and \( 0 \leq \mu < 2^L \), the disjoint groups are denoted by \( \ell_{\mu}^{(L)} \). The groups’ center coordinates are indicated by \( r_{\ell,\mu}^{(L)} \), and the group-internal coordinates are \( r_{\ell,vir,v}^{(L)} = r_{\ell,vir,v} - r_{\ell, c,vir,\mu}^{(L)} \), \( 0 \leq v' < 2^{-L} N \). They are identical for all groups \( \ell_{\mu}^{(L)} \).

For a given group of virtual transducer elements \( \ell_{\mu}^{(L)} \), the groups of lattice points can be classified into well-separated groups and direct neighbors based on their center coordinates’ distances. The distance between two groups is defined as

\[
d_{\mu,m,n}^{(L)} = \| d_{\mu,m,n}^{(L)} \|_2, \tag{8}
\]

where

\[
d_{\mu,m,n}^{(L)} = r_{\ell, lat,m,n}^{(L)} - r_{\ell, c, lat,m,n}^{(L)}. \tag{9}
\]

The set of indices associated with all groups of lattice points is

\[
\mathcal{G}^{(L)} = \{ (m,n) \in \mathbb{N}_0^2 : 0 \leq m, n < 2^L \}. \tag{10}
\]

The set of indices associated with well-separated groups is

\[
\mathcal{W}_{\mu}^{(L)} = \{ (m,n) \in \mathcal{G}^{(L)} : d_{\mu,m,n}^{(L)} > d^{(L)} \}. \tag{11}
\]

The set of direct neighbors is the complement of \( \mathcal{W}_{\mu}^{(L)} \) with respect to \( \mathcal{G}^{(L)} \)

\[
\mathcal{W}_{\mu}^{(L)} = \mathcal{G}^{(L)} \setminus \mathcal{W}_{\mu}^{(L)}. \tag{12}
\]

#### B. Forward Scattering Operation for Direct Neighbors

For \( (m,n) \in \mathcal{W}_{\mu}^{(L)} \), the system of linear equations \( \{4\} \) cannot be treated by the FMM and is partially written as

\[
p_{sc}^{\ell_{\ell',\mu,m,n}}(e_{\vartheta,d}) = G_{l,\ell',\mu,m,n}(e_{\vartheta,d}) \gamma_{n, BP,m,n}. \tag{13}
\]
where we defined the $2^{-L} N \times (2^{-L} N)^2$ direct scattering matrix
\[
\{ G_{l,\mu,m,n}(e_{\theta,d}) \}_{\nu',\nu} = (k_0 c)^2 P_l(e_{\theta,d}) \mathcal{e}^{-j k_0 e_{\theta,d} \cdot (\hat{r}_{l,\mu,m,n} + \hat{r}_{\nu'})} 
\times g_l(d_{\mu,m,n} - (\hat{r}_{\nu'} - \hat{r}_{\nu})),
\]
and the $(2^{-L} N)^2 \times 1$ vector
\[
\{ \gamma_{\kappa,BP,m,n} \}_\nu = \gamma_{\kappa,BP}(\hat{r}_{l,\mu,m,n} + \hat{r}_{\nu}).
\]

C. Forward Scattering Operation for Well-Separated Groups

For $(m, n) \in \mathcal{W}_\mu(L)$, the Green’s function’s truncated multipole expansion may be used to modify the system of linear equations (4). The accuracy of the expansion is governed by the parameter $N_{e_{\theta,d}}(L)$. Adapting the result given in [10, (24)] for three spatial dimensions yields
\[
N_{e_{\theta,d}}(L) = 2 [k_l d(L) + 5 \ln(k_l d(L) + \pi)] + 1.
\]

In this case, we may write the scattered acoustic pressure received by the virtual elements in $\mathcal{W}_\mu(L)$ and originating from the point scatterers in the well-separated group $\mathcal{L}_{m,n}$ as

\[
\mathbf{p}_{l,\mu,m,n}(e_{\theta,d}) \approx \phi_{l,m,n}(e_{\theta,d}) \mathbf{D}_l \mathbf{T}_{l,\mu,m,n} \mathbf{A}_l(e_{\theta,d}) 
\times \gamma_{\kappa,BP,m,n},
\]

where $\mathbf{A}_l(e_{\theta,d})$ is the $N_{e_{\theta,d}}(L) \times (2^{-L} N)^2$ aggregation matrix, $\mathbf{T}_{l,\mu,m,n}$ is the $N_{e_{\theta,d}}(L) \times N_{e_{\theta,d}}(L)$ diagonal translation matrix, $\mathbf{D}_l$ is the $2^{-L} N \times N_{e_{\theta,d}}(L)$ disaggregation matrix, and $\phi_{l,m,n}(e_{\theta,d})$ is a scalar factor.

D. Complete Forward Scattering Operation

Considering all groups of lattice points $\mathcal{L}_{m,n}$, $0 \leq m, n < 2^L$, the scattered acoustic pressure received by the virtual elements in $\mathcal{W}_\mu(L)$ is

\[
\mathbf{p}_{l,\mu}(e_{\theta,d}) = \sum_{(m,n) \in \mathcal{W}_\mu(L)} \mathbf{p}_{l,\mu,m,n}(e_{\theta,d}) 
+ \sum_{(m,n) \in \mathcal{W}_\mu(L)} \mathbf{p}_{l,\mu,m,n}(e_{\theta,d}) 
\approx \sum_{(m,n) \in \mathcal{W}_\mu(L)} \mathbf{G}_{l,\mu,m,n}(e_{\theta,d}) \gamma_{\kappa,BP,m,n} 
+ \mathbf{D}_l \sum_{(m,n) \in \mathcal{W}_\mu(L)} \phi_{l,m,n}(e_{\theta,d}) \mathbf{T}_{l,\mu,m,n} 
\times \mathbf{A}_l(e_{\theta,d}) \gamma_{\kappa,BP,m,n}.
\]

This result can be applied to all groups $\mathcal{W}_\mu(L)$, $0 \leq \mu < 2^L$, for all frequencies $f_1$, $0 \leq l < N_f$, and for all directions $e_{\theta,d}$, $0 \leq d < N_\theta$, of the incident plane waves (5).

IV. MEMORY REQUIREMENTS AND COMPUTATIONAL COSTS

The expressions (7) and (9) significantly differ in the amount of memory required to store the data structures and in the computational costs associated with their evaluation. For the following results, we assumed the parameters $L_0 = 9$.

### Fig. 2. Relative amounts of memory required by the FMM of level $L$ for $3 \leq L \leq 7$. The amounts were normalized by (10). The absolute amounts printed next to the bars’ tops assume 64 bit double-precision floating-point arithmetic for real-valued variables.

\[(N = 512 = 2^{10}, \varsigma = 76.2 \mu m, z_0 = 2^{-1} c, N_f = 230\] in a bandwidth from 2.6 MHz to 5.4 MHz, and $N_\theta = 1$, which correspond to a problem of typical size. Moreover, we assumed 64 bit double-precision floating-point arithmetic for real-valued variables, i.e. the amount of memory allocated for a complex-valued variable is $w_c = 16 B$.

A. Conventional Matrix-Vector Product

The amount of memory required for the storage of the matrix $\mathbf{G}$ in (7) is

\[M_{\text{conv}}(N_\theta) = N_\theta N_f N^3 w_c = 460 \text{ GiB}.\]  

The number of complex-valued multiplications in the conventional matrix-vector product is

\[N_{\text{mul,conv}}(N_\theta) = N_\theta N_f N^3 \approx 3.087 \times 10^{10}.\]

B. Fast Multipole Method

The amount of memory occupied by the data structures in the FMM of level $L$ can be computed in an analog manner. Besides the storage of the matrices $\mathbf{G}_{l,\mu,m,n}(e_{\theta,d})$, $\mathbf{A}_l(e_{\theta,d})$, $\mathbf{T}_{l,\mu,m,n}$, $\mathbf{D}_l$, and the scalar factors $\phi_{l,m,n}(e_{\theta,d})$, intermediate results have to be stored. We assumed that the results of the aggregation step, the results of the translation step, and the results of the disaggregation step are stored.

Fig. 3 depicts the resulting amounts of memory required by the data structures in the FMM for the levels $3 \leq L \leq 7$. These amounts were normalized by (10). The absolute amounts are printed next to the bars’ tops. For $L = 5$, the FMM consumes less than 0.25% of the memory required for the conventional matrix-vector product!

Fig. 3 depicts the number of complex-valued multiplications in the FMM for the levels $3 \leq L \leq 7$. These numbers were normalized by (11). The absolute numbers are printed next to the bars’ tops. For $L = 5$, the FMM computes less than 24% of the number of complex-valued multiplications induced by the conventional matrix-vector product!
The formulation of pulse-echo ultrasound imaging (UI) as a linear inverse scattering problem (ISP) in the temporal frequency domain enables the usage of realistic physical models in the process of image reconstruction [3], [4]. In combination with novel regularization procedures, e.g., compressed sensing (CS) [5], this formulation significantly improves image quality [6], [7]. However, major challenges of these procedures are the memory consumption and computational costs associated with the conventional matrix-vector product.

In this contribution, we analyzed the fast multipole method (FMM) as a tool to meet these challenges. For an example of typical size, we demonstrated that the FMM consumes less than 0.25% of the memory required for the conventional matrix-vector product.

Additionally, the FMM computes less than 24% of the number of complex-valued multiplications induced by the conventional matrix-vector product.

The FMM thus enables the numerical solution of the regularized linear ISP, e.g., by CS [5], on standard personal computers. It significantly improves the applicability of inverse scattering strategies in practical UI.

**REFERENCES**


